4.1

$$\int_a^b e^x\,dx = \lim_{\substack{n\to\infty\\ \Delta x\to 0}} \sum_{k=1}^{n} f(x_k)\Delta x \qquad \text{mit } \Delta x = \frac{b-a}{n} \text{ und } x_k = a+k\Delta x$$

$$\sum_{k=1}^{n} f(x_k)\Delta x = \sum_{k=1}^{n} e^{x_k}\Delta x = \sum_{k=1}^{n} e^{a+k\Delta x}\Delta x = \sum_{k=1}^{n} e^{a}e^{k\Delta x}\Delta x$$

$$= e^a\Delta x\sum_{k=1}^{n} e^{k\Delta x} = e^a\Delta x\sum_{k=1}^{n}\left(e^{\Delta x}\right)^k = e^a\Delta x\sum_{k=1}^{n} q^k \quad \text{mit } q=e^{\Delta x}$$

$$= e^a\Delta x\Big(\underbrace{\sum_{k=0}^{n-1} q^k}_{\frac{1-q^n}{1-q}} + q^n - 1\Big) = e^a\Delta x\left(\frac{1-q^n}{1-q} + q^n - 1\right)$$

$$= e^a\Delta x\left(\frac{q^n-1}{q-1} + q^n - 1\right) = e^a\Delta x\,(q^n-1)\left(\frac{1}{q-1}+1\right)$$

$$= e^a\Delta x\left(\left(e^{\Delta x}\right)^n - 1\right)\left(\frac{1}{e^{\Delta x}-1}+1\right) = e^a\Delta x\left(e^{n\Delta x}-1\right)\left(\frac{1}{e^{\Delta x}-1}+1\right)$$

$$= e^a\left(e^{n\Delta x}-1\right)\left(\frac{\Delta x}{e^{\Delta x}-1}+\Delta x\right) = \left(e^a e^{n\Delta x} - e^a\right)\left(\frac{\Delta x}{e^{\Delta x}-1}+\Delta x\right)$$

$$= \left(e^{a+n\Delta x} - e^a\right)\left(\frac{\Delta x}{e^{\Delta x}-1}+\Delta x\right) = \left(e^b - e^a\right)\left(\frac{\Delta x}{e^{\Delta x}-1}+\Delta x\right)$$

$$\lim_{\substack{n\to\infty\\ \Delta x\to 0}} \sum_{k=1}^{n} f(x_k)\Delta x = \lim_{\Delta x\to 0}\left(e^b-e^a\right)\left(\frac{\Delta x}{e^{\Delta x}-1}+\Delta x\right)$$

$$= \left(e^b-e^a\right)\lim_{\Delta x\to 0}\left(\frac{\Delta x}{e^{\Delta x}-1}+\Delta x\right) = e^b - e^a$$

4.2 a)

$$\sin(x^3) \text{ ungerade} \Rightarrow \int_{-\sqrt{2}}^{\sqrt{2}} \sin(x^3)\,dx = 0$$

b)

$$\tanh(x+\sin x) \text{ ungerade} \Rightarrow \int_{-\pi}^{\pi} \tanh(x+\sin x)\,dx = 0$$

4.3 a)

$$\int \cos\left(\tfrac{1}{2}x+\pi\right)dx = -\int \cos\frac{x}{2}\,dx \overset{(4.24)}{=} -2\sin\frac{x}{2}$$

b) $$\int \frac{4x}{2+x^2}\,dx = 2\int \frac{2x}{2+x^2}\,dx \overset{(4.18)}{=} 2\ln(2+x^2)$$

c) $$\int \frac{3\cos x}{2+\sin x}\,dx = 3\int \frac{\cos x}{2+\sin x}\,dx \overset{(4.18)}{=} 3\ln(2+\sin x)$$

d) $$\int \frac{4}{4+x^2}\,dx = \int \frac{4}{4\left(1+\left(\frac{x}{2}\right)^2\right)}\,dx = \int \frac{1}{1+\left(\frac{x}{2}\right)^2}\,dx$$
$$\overset{(4.24)}{=} 2\arctan\frac{x}{2}$$

e) $$\frac{4x}{3+2x} = 2\,\frac{2x}{3+2x} = 2\,\frac{3+2x-3}{3+2x} = 2\left(1-\frac{3}{3+2x}\right)$$
$$= 2 - 3\,\frac{2}{3+2x}$$
$$\int \frac{4x}{3+2x}\,dx = \int 2-3\,\frac{2}{3+2x}\,dx = 2\int dx - 3\int \frac{2}{3+2x}\,dx$$
$$= 2x - 3\ln|3+2x|$$

4.4 a) $$\int \sqrt{x}\ln x\,dx = \int x^{\frac{1}{2}}\ln x\,dx = \int \left(\frac{2}{3}x^{\frac{3}{2}}\right)'\ln x\,dx$$
$$= \frac{2}{3}x^{\frac{3}{2}}\ln x - \int \frac{2}{3}x^{\frac{3}{2}}(\ln x)'\,dx$$
$$= \frac{2}{3}x^{\frac{3}{2}}\ln x - \int \frac{2}{3}x^{\frac{3}{2}}\frac{1}{x}\,dx = \frac{2}{3}x^{\frac{3}{2}}\ln x - \frac{2}{3}\int x^{\frac{1}{2}}\,dx$$
$$= \frac{2}{3}x^{\frac{3}{2}}\ln x - \frac{2}{3}\,\frac{2}{3}x^{\frac{3}{2}} = \frac{2}{3}\sqrt{x^3}\ln x - \frac{4}{9}\sqrt{x^3}$$

b) $$\int x^3 e^x\,dx = \int x^3(e^x)'\,dx = x^3e^x - \int (x^3)'e^x\,dx$$
$$= x^3e^x - \int 3x^2e^x\,dx = x^3e^x - \int 3x^2(e^x)'\,dx$$
$$= x^3e^x - \left(3x^2e^x - \int (3x^2)'e^x\,dx\right) = x^3e^x - 3x^2e^x + \int 6x\,e^x\,dx$$
$$= x^3e^x - 3x^2e^x + \int 6x(e^x)'\,dx = x^3e^x - 3x^2e^x + 6x\,e^x - \int (6x)'e^x\,dx$$
$$= x^3e^x - 3x^2e^x + 6x\,e^x - 6\int e^x\,dx = x^3e^x - 3x^2e^x + 6x\,e^x - 6e^x$$
$$= (x^3 - 3x^2 + 6x - 6)\,e^x$$

c)

$$\int \frac{\ln x}{x}\,dx = \int \ln x (\ln x)'\,dx = \ln x \ln x - \int (\ln x)' \ln x\,dx$$

$$= (\ln x)^2 - \int \frac{\ln x}{x}\,dx$$

$$2\int \frac{\ln x}{x}\,dx = (\ln x)^2 \qquad \int \frac{\ln x}{x}\,dx = \frac{1}{2}(\ln x)^2$$

d)

$$\int \sin^2 x\,dx = -\int \sin x (\cos x)'\,dx = -\left(\sin x \cos x - \int (\sin x)' \cos x\,dx\right)$$

$$= -\sin x \cos x + \int \cos^2 x\,dx = -\sin x \cos x + \int 1 - \sin^2 x\,dx$$

$$= -\sin x \cos x + \int dx - \int \sin^2 x\,dx = -\sin x \cos x + x - \int \sin^2 x\,dx$$

$$2\int \sin^2 x\,dx = x - \sin x \cos x \qquad \int \sin^2 x\,dx = \frac{x}{2} - \frac{1}{2}\sin x \cos x$$

e)

$$\int x(\ln x)^2\,dx = \int \left(\tfrac{1}{2}x^2\right)'(\ln x)^2\,dx = \tfrac{1}{2}x^2(\ln x)^2 - \int \tfrac{1}{2}x^2\, 2\ln x \cdot \tfrac{1}{x}\,dx$$

$$= \tfrac{1}{2}x^2(\ln x)^2 - \int x \ln x\,dx = \tfrac{1}{2}x^2(\ln x)^2 - \int \left(\tfrac{1}{2}x^2\right)' \ln x\,dx$$

$$= \tfrac{1}{2}x^2(\ln x)^2 - \left(\tfrac{1}{2}x^2 \ln x - \int \tfrac{1}{2}x^2\, \tfrac{1}{x}\,dx\right)$$

$$= \tfrac{1}{2}x^2(\ln x)^2 - \tfrac{1}{2}x^2 \ln x + \int \tfrac{1}{2}x\,dx$$

$$= \tfrac{1}{2}x^2(\ln x)^2 - \tfrac{1}{2}x^2 \ln x + \tfrac{1}{4}x^2 = \tfrac{1}{2}x^2\left[(\ln x)^2 - \ln x + \tfrac{1}{2}\right]$$

f)

$$\int e^x \sin x\,dx = \int (e^x)' \sin x\,dx = e^x \sin x - \int e^x (\sin x)'\,dx$$

$$= e^x \sin x - \int e^x \cos x\,dx = e^x \sin x - \int (e^x)' \cos x\,dx$$

$$= e^x \sin x - \left(e^x \cos x - \int e^x (\cos x)'\,dx\right)$$

$$= e^x \sin x - e^x \cos x - \int e^x \sin x\,dx$$

$$2\int e^x \sin x\,dx = e^x \sin x - e^x \cos x$$

$$\int e^x \sin x\,dx = \tfrac{1}{2}\left(e^x \sin x - e^x \cos x\right) = \tfrac{1}{2}e^x(\sin x - \cos x)$$

g)

$$\int x^2(\ln(x^2))^2\,dx = \int(\tfrac{1}{3}x^3)'(\ln(x^2))^2\,dx$$

$$= \tfrac{1}{3}x^3(\ln(x^2))^2 - \int \tfrac{1}{3}x^3\,2\ln(x^2)\,\tfrac{1}{x^2}\,2x\,dx$$

$$= \tfrac{1}{3}x^3(\ln(x^2))^2 - \int \tfrac{4}{3}x^2\ln(x^2)\,dx$$

$$= \tfrac{1}{3}x^3(\ln(x^2))^2 - \int(\tfrac{4}{9}x^3)'\ln(x^2)\,dx$$

$$= \tfrac{1}{3}x^3(\ln(x^2))^2 - \left(\tfrac{4}{9}x^3\ln(x^2) - \int \tfrac{4}{9}x^3\,\tfrac{1}{x^2}\,2x\,dx\right)$$

$$= \tfrac{1}{3}x^3(\ln(x^2))^2 - \tfrac{4}{9}x^3\ln(x^2) + \tfrac{8}{9}\int x^2\,dx$$

$$= \tfrac{1}{3}x^3(\ln(x^2))^2 - \tfrac{4}{9}x^3\ln(x^2) + \tfrac{8}{27}x^3$$

$$= \tfrac{1}{3}x^3\left[(\ln(x^2))^2 - \tfrac{4}{3}\ln(x^2) + \tfrac{8}{9}\right]$$

4.5 a)

$$\int x e^{x^2}\,dx \qquad x^2 = u \qquad \frac{du}{dx} = 2x \qquad dx = \frac{1}{2x}\,du$$

$$= \int x e^u \frac{1}{2x}\,du = \frac{1}{2}\int e^u\,du = \frac{1}{2}e^u = \frac{1}{2}e^{x^2}$$

b)

$$\int \frac{\ln x}{x(1+\ln x)}\,dx \qquad 1+\ln x = u \qquad \frac{du}{dx} = \frac{1}{x} \qquad dx = x\,du$$

$$\ln x = u-1$$

$$= \int \frac{u-1}{x\,u}\,x\,du = \int \frac{u-1}{u}\,du = \int 1 - \frac{1}{u}\,du$$

$$= u - \ln|u| = 1+\ln x - \ln|1+\ln x|$$

c)

$$\int \frac{e^x}{\sqrt{1+e^x}}\,dx \qquad \sqrt{1+e^x} = u \qquad \frac{du}{dx} = \frac{1}{2\sqrt{1+e^x}}\,e^x = \frac{e^x}{2u}$$

$$dx = \frac{2u}{e^x}\,du$$

$$= \int \frac{e^x}{u}\,\frac{2u}{e^x}\,du = 2\int du = 2u = 2\sqrt{1+e^x}$$

d) $\int \frac{1}{x^3} e^{-\frac{1}{x}} dx \qquad -\frac{1}{x} = u \quad \frac{du}{dx} = \frac{1}{x^2} \quad dx = x^2 du$

$= \int \frac{1}{x^3} e^u x^2 du = \int \frac{1}{x} e^u du = -\int u e^u du$

$= -\int u (e^u)' du = -\left(u e^u - \int e^u du\right) = -u e^u + \int e^u du$

$= -u e^u + e^u = \frac{1}{x} e^{-\frac{1}{x}} + e^{-\frac{1}{x}}$

e) $\int \frac{1}{\sqrt{x^2+4}} dx \qquad x = 2\sinh u \quad \frac{dx}{du} = 2\cosh u \quad dx = 2\cosh u \cdot du$

$\sinh u = \frac{x}{2} \quad u = \operatorname{arsinh} \frac{x}{2}$

$= \int \frac{1}{\sqrt{(2\sinh u)^2+4}} 2\cosh u \cdot du = \int \frac{2\cosh u}{\sqrt{4\sinh^2 u+4}} du$

$= \int \frac{\cosh u}{\sqrt{\sinh^2 u+1}} du = \int \frac{\cosh u}{\sqrt{\cosh^2 u}} du = \int \frac{\cosh u}{\cosh u} du$

$= \int du = u = \operatorname{arsinh} \frac{x}{2}$

4.6 a) $\int x^5 e^{x^2} dx \qquad x^2 = u \quad \frac{du}{dx} = 2x \quad dx = \frac{1}{2x} du$

$= \int x^5 e^u \frac{1}{2x} du = \frac{1}{2} \int x^4 e^u du = \frac{1}{2} \int u^2 e^u du$

$= \frac{1}{2} \int u^2 (e^u)' du = \frac{1}{2}\left(u^2 e^u - \int 2u e^u du\right)$

$= \frac{1}{2} u^2 e^u - \int u (e^u)' du = \frac{1}{2} u^2 e^u - \left(u e^u - \int e^u du\right)$

$= \frac{1}{2} u^2 e^u - u e^u + e^u = e^u\left(\frac{1}{2}u^2 - u + 1\right)$

$= e^{x^2}\left(\frac{1}{2}x^4 - x^2 + 1\right)$

b) $\int \ln(1-x) dx \qquad 1-x = u \quad \frac{du}{dx} = -1 \quad dx = -du$

$= -\int \ln u \, du = -\int u' \ln u \, du = -\left(u \ln u - \int u \frac{1}{u} du\right)$

$= -u \ln u + \int du = -u \ln u + u = -(1-x)\ln(1-x) + 1 - x$

c)

$$\int \ln(1+x)\,dx \qquad 1+x=u \quad \frac{du}{dx}=1 \quad dx=du$$

$$=\int \ln u\,du = \int u'\ln u\,du = u\ln u-\int u\,\frac{1}{u}\,du = u\ln u-\int du$$

$$= u\ln u - u = (1+x)\ln(1+x)-(1+x)$$

d)

$$\int \ln(1-x^2)\,dx = \int \ln[(1-x)(1+x)]\,dx = \int \ln(1-x)+\ln(1+x)\,dx$$

$$=\int \ln(1-x)\,dx+\int \ln(1+x)\,dx = -(1-x)\ln(1-x)+1-x+(1+x)\ln(1+x)-(1+x)$$

$$= (1+x)\ln(1+x)-(1-x)\ln(1-x)-2x$$

e)

$$\int \cos\ln x\,dx \qquad u=\ln x \quad \frac{du}{dx}=\frac{1}{x} \quad dx = x\,du$$

$$=\int \cos u\cdot x\,du \qquad x=e^u$$

$$=\int e^u\cos u\,du = \int (e^u)'\cos u\,du = e^u\cos u-\int e^u(\cos u)'\,du$$

$$= e^u\cos u+\int e^u\sin u\,du = e^u\cos u+\int (e^u)'\sin u\,du$$

$$= e^u\cos u + e^u\sin u-\int e^u(\sin u)'\,du$$

$$= e^u\cos u+e^u\sin u-\int e^u\cos u\,du$$

$$2\int e^u\cos u\,du = e^u\cos u+e^u\sin u$$

$$\int e^u\cos u\,du = \frac{1}{2}e^u(\cos u+\sin u)$$

$$\int \cos\ln x\,dx = \int e^u\cos u\,du = \frac{1}{2}e^u(\cos u+\sin u)$$

$$= \frac{1}{2}e^{\ln x}(\cos\ln x+\sin\ln x) = \frac{1}{2}x(\cos\ln x+\sin\ln x)$$

f)

$$\int \cosh\ln x\,dx = \int \frac{1}{2}\left(e^{\ln x}+e^{-\ln x}\right)dx = \int \frac{1}{2}\left(e^{\ln x}+\frac{1}{e^{\ln x}}\right)dx$$

$$=\int \frac{1}{2}\left(x+\frac{1}{x}\right)dx = \int \frac{1}{2}x+\frac{1}{2}\frac{1}{x}\,dx$$

$$= \frac{1}{4}x^2+\frac{1}{2}\ln x$$

4.7 a)

$$\frac{3x^3+3x+2}{x^4-1} \overset{\text{Partialbruch-}}{\underset{\text{zerlegung}}{=}} \frac{2}{x-1}+\frac{1}{x+1}-\frac{1}{x^2+1}$$

$$\int \frac{3x^3+3x+2}{x^4-1}\,dx = \int \frac{2}{x-1}+\frac{1}{x+1}-\frac{1}{x^2+1}\,dx$$

$$= 2\ln|x-1| + \ln|x+1| - \arctan x$$

b)

$$\frac{2x^2+2x+2}{x^4-2x^3+2x^2-2x+1} \overset{\text{Partialbruch-}}{\underset{\text{zerlegung}}{=}} \frac{3}{(x-1)^2}-\frac{1}{x^2+1}$$

$$\int \frac{2x^2+2x+2}{x^4-2x^3+2x^2-2x+1}\,dx = \int \frac{3}{(x-1)^2}-\frac{1}{x^2+1}\,dx$$

$$= -\frac{3}{x-1} - \arctan x$$

4.8 a)

$$\int \sinh x \cdot \ln \cosh x \,dx \qquad \cosh x = u \quad \frac{du}{dx} = \sinh x \quad dx = \frac{1}{\sinh x}\,du$$

$$= \int \sinh x \ln u \cdot \frac{1}{\sinh x}\,du = \int \ln u\,du = u\ln u - u$$

$$= \cosh x \ln \cosh x - \cosh x$$

$$\int \sinh x \ln\cosh x\,dx = \int (\cosh x)' \ln \cosh x\,dx$$

$$= \cosh x \ln \cosh x - \int \cosh x\,(\ln \cosh x)'\,dx$$

$$= \cosh x \ln \cosh x - \int \cosh x \frac{1}{\cosh x} \sinh x\,dx$$

$$= \cosh x \ln \cosh x - \int \sinh x\,dx = \cosh x \ln \cosh x - \cosh x$$

b)

$$\int (2x+e^x)\ln(x^2+e^x)\,dx \qquad u = x^2+e^x \quad \frac{du}{dx} = 2x+e^x \quad dx = \frac{1}{2x+e^x}\,du$$

$$= \int (2x+e^x)\ln u \,\frac{1}{2x+e^x}\,du = \int \ln u\,du = u\ln u - u$$

$$= (x^2+e^x)\ln(x^2+e^x) - (x^2+e^x)$$

$$\int (2x+e^x)\ln(x^2+e^x)\,dx = \int (x^2+e^x)'\ln(x^2+e^x)\,dx$$

$$= (x^2+e^x)\ln(x^2+e^x) - \int (x^2+e^x)\left(\ln(x^2+e^x)\right)'\,dx$$

$$= (x^2+e^x)\ln(x^2+e^x) - \int (x^2+e^x)\,\frac{1}{x^2+e^x}(2x+e^x)\,dx$$

$$= (x^2+e^x)\ln(x^2+e^x) - \int 2x+e^x\,dx = (x^2+e^x)\ln(x^2+e^x) - (x^2+e^x)$$

c)

$$\int \frac{(\ln x)^2}{x}\,dx \qquad u = \ln x \quad \frac{du}{dx} = \frac{1}{x} \quad dx = x\,du$$

$$= \int \frac{u^2}{x}x\,du = \int u^2\,du = \frac{1}{3}u^3 = \frac{1}{3}(\ln x)^3$$

$$\int \frac{(\ln x)^2}{x}\,dx = \int (\ln x)'(\ln x)^2\,dx = \ln x\,(\ln x)^2 - \int \ln x\left((\ln x)^2\right)'\,dx$$

$$= (\ln x)^3 - \int \ln x\cdot 2\ln x\cdot\frac{1}{x}\,dx = (\ln x)^3 - 2\int\frac{(\ln x)^2}{x}\,dx$$

$$\Rightarrow 3\int \frac{(\ln x)^2}{x}\,dx = (\ln x)^3 \qquad \int\frac{(\ln x)^2}{x}\,dx = \frac{1}{3}(\ln x)^3$$

4.9 a)

$$\int_0^2 \frac{2x}{1+\frac{1}{4}x^2}\,dx = \int_0^2 \frac{2x}{\frac{1}{4}(4+x^2)}\,dx = 4\int_0^2 \frac{2x}{4+x^2}\,dx$$

$$= 4\left[\ln(4+x^2)\right]_0^2 = 4(\ln 8 - \ln 4) = 4\ln\frac{8}{4} = 4\ln 2$$

b) $$\int_0^{\frac{1}{4}\pi^2} \cos\sqrt{x}\,dx \qquad u=\sqrt{x} \quad \frac{du}{dx}=\frac{1}{2\sqrt{x}} \quad dx = 2\sqrt{x}\,du = 2u\,du$$

$$x_1 = 0 \quad u_1 = \sqrt{x_1} = \sqrt{0} = 0$$

$$x_2 = \frac{1}{4}\pi^2 \quad u_2 = \sqrt{x_2} = \sqrt{\frac{1}{4}\pi^2} = \frac{\pi}{2}$$

$$= \int_{u_1}^{u_2} \cos u \cdot 2u\,du = 2\int_0^{\frac{\pi}{2}} u\cos u\,du = 2\int_0^{\frac{\pi}{2}} u(\sin u)'\,du$$

$$= 2\left(\left[u\sin u\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin u\,du\right) = 2\left(\left[u\sin u\right]_0^{\frac{\pi}{2}} + \left[\cos u\right]_0^{\frac{\pi}{2}}\right)$$

$$= 2\left(\frac{\pi}{2} - 1\right) = \pi - 2$$

c) $$\int_0^{2\pi} \cos^2 x\,dx = \int_0^{2\pi} \cos x \cdot \cos x\,dx = \int_0^{2\pi} (\sin x)'\cos x\,dx$$

$$= \left[\sin x\cos x\right]_0^{2\pi} - \int_0^{2\pi} \sin x(\cos x)'\,dx = \int_0^{2\pi} \sin^2 x\,dx = \int_0^{2\pi} 1 - \cos^2 x\,dx$$

$$= \int_0^{2\pi} dx - \int_0^{2\pi} \cos^2 x\,dx = 2\pi - \int_0^{2\pi} \cos^2 x\,dx$$

$$\Rightarrow 2\int_0^{2\pi} \cos^2 x\,dx = 2\pi \qquad \int_0^{2\pi} \cos^2 x\,dx = \pi$$

4.10 a) $$\int x^2 e^{-x}\,dx = \int x^2(-e^{-x})'\,dx = -x^2e^{-x} + \int (x^2)' e^{-x}\,dx$$

$$= -x^2e^{-x} + \int 2x e^{-x}\,dx = -x^2e^{-x} + \int 2x(-e^{-x})'\,dx$$

$$= -x^2e^{-x} - 2xe^{-x} + 2\int e^{-x}\,dx$$

$$= -x^2e^{-x} - 2xe^{-x} - 2e^{-x} = -e^{-x}(x^2+2x+2) = -\frac{x^2+2x+2}{e^x}$$

$$\int_0^{\infty} x^2e^{-x}\,dx = \lim_{\lambda\to\infty}\int_0^{\lambda} x^2e^{-x}\,dx = \lim_{\lambda\to\infty}\left[-\frac{x^2+2x+2}{e^x}\right]_0^{\lambda}$$

$$= \lim_{\lambda\to\infty}\left(-\frac{\lambda^2+2\lambda+2}{e^\lambda} + 2\right) = -\lim_{\lambda\to\infty}\frac{\lambda^2+2\lambda+2}{e^\lambda} + 2$$

$$\overset{\text{L'H.}}{=} -\lim_{\lambda\to\infty}\frac{2\lambda+2}{e^\lambda} + 2 \overset{\text{L'H.}}{=} -\lim_{\lambda\to\infty}\frac{2}{e^\lambda} + 2 = 2$$

b)

$$\int x^3 e^{-x^2} dx \qquad -x^2 = u \quad \frac{du}{dx} = -2x \qquad dx = -\frac{1}{2x} du$$

$$= \int x^3 e^u \left(-\frac{1}{2x}\right) du = \frac{1}{2}\int -x^2 e^u du = \frac{1}{2}\int u e^u du$$

$$= \frac{1}{2}\int u (e^u)' du = \frac{1}{2}\left(u e^u - \int e^u du\right) = \frac{1}{2}\left(u e^u - e^u\right)$$

$$= \frac{1}{2}\left(-x^2 e^{-x^2} - e^{-x^2}\right) = -\frac{1}{2} e^{-x^2}\left(1+x^2\right) = -\frac{1}{2}\,\frac{1+x^2}{e^{x^2}}$$

$$\int_0^\infty x^3 e^{-x^2} dx = \lim_{\lambda\to\infty}\int_0^\lambda x^3 e^{-x^2} dx = \lim_{\lambda\to\infty}\left[-\frac{1}{2}\frac{1+x^2}{e^{x^2}}\right]_0^\lambda$$

$$= \lim_{\lambda\to\infty}\left(-\frac{1}{2}\frac{1+\lambda^2}{e^{\lambda^2}} + \frac{1}{2}\right) = -\frac{1}{2}\lim_{\lambda\to\infty}\frac{1+\lambda^2}{e^{\lambda^2}} + \frac{1}{2} = -\frac{1}{2}\lim_{\lambda\to\infty}\frac{2\lambda}{e^{\lambda^2}2\lambda} + \frac{1}{2}$$

$$= -\frac{1}{2}\lim_{\lambda\to\infty}\frac{1}{e^{\lambda^2}} + \frac{1}{2} = \frac{1}{2}$$

c)

$$\int e^{-\sqrt{x}} dx \qquad -\sqrt{x} = u \quad \frac{du}{dx} = -\frac{1}{2\sqrt{x}} \quad dx = -2\sqrt{x}\,du = 2u\,du$$

$$= 2\int u e^u du = 2\int u (e^u)' du = 2\left(u e^u - \int e^u du\right) = 2\left(u e^u - e^u\right)$$

$$= 2e^u(u-1) = 2e^{-\sqrt{x}}\left(-\sqrt{x}-1\right) = -2e^{-\sqrt{x}}\left(\sqrt{x}+1\right)$$

$$\int_0^\infty e^{-\sqrt{x}} dx = \lim_{\lambda\to\infty}\int_0^\lambda e^{-\sqrt{x}} dx = \lim_{\lambda\to\infty}\left[-2e^{-\sqrt{x}}(\sqrt{x}+1)\right]_0^\lambda$$

$$= \lim_{\lambda\to\infty}\left(-2e^{-\sqrt{\lambda}}(\sqrt{\lambda}+1) + 2\right) = -2\lim_{\lambda\to\infty}\frac{\sqrt{\lambda}+1}{e^{\sqrt{\lambda}}} + 2$$

$$\overset{\text{L'H.}}{=} -2\lim_{\lambda\to\infty}\frac{\frac{1}{2\sqrt{\lambda}}}{e^{\sqrt{\lambda}}\frac{1}{2\sqrt{\lambda}}} + 2 = -2\lim_{\lambda\to\infty}\frac{1}{e^{\sqrt{\lambda}}} + 2 = 2$$

d)

$$\int \frac{x}{\cosh^2 x}\,dx = \int x \cdot \frac{1}{\cosh^2 x}\,dx = \int x\,(\tanh x)'\,dx$$

$$= x\tanh x - \int \tanh x\,dx = x\tanh x - \int \frac{\sinh x}{\cosh x}\,dx$$

$$= x\tanh x - \ln\cosh x$$

$$\int_0^\infty \frac{x}{\cosh^2 x}\,dx = \lim_{\lambda\to\infty}\int_0^\lambda \frac{x}{\cosh^2 x}\,dx = \lim_{\lambda\to\infty}\Big[x\tanh x - \ln\cosh x\Big]_0^\lambda$$

$$= \lim_{\lambda\to\infty}(\lambda\tanh\lambda - \ln\cosh\lambda) = \ln 2 \qquad \text{s. Aufgabe 3.13 k)}$$

e)

$$\int \frac{1}{x^2} e^{-\frac{1}{x}}\,dx \qquad u = -\frac{1}{x} \quad \frac{du}{dx} = \frac{1}{x^2} \quad dx = x^2\,du$$

$$= \int \frac{1}{x^2} e^u x^2\,du = \int e^u\,du = e^u = e^{-\frac{1}{x}}$$

$$\int_0^\infty \frac{1}{x^2} e^{-\frac{1}{x}}\,dx = \lim_{\varepsilon\to 0}\lim_{\lambda\to\infty}\int_\varepsilon^\lambda \frac{1}{x^2} e^{-\frac{1}{x}}\,dx = \lim_{\varepsilon\to 0}\lim_{\lambda\to\infty}\Big[e^{-\frac{1}{x}}\Big]_\varepsilon^\lambda$$

$$= \lim_{\varepsilon\to 0}\lim_{\lambda\to\infty}\left(e^{-\frac{1}{\lambda}} - e^{-\frac{1}{\varepsilon}}\right) = \lim_{\lambda\to\infty} e^{-\frac{1}{\lambda}} - \lim_{\varepsilon\to 0} e^{-\frac{1}{\varepsilon}}$$

$$= \lim_{\lambda\to\infty}\frac{1}{e^{\frac{1}{\lambda}}} - \lim_{\varepsilon\to 0}\frac{1}{e^{\frac{1}{\varepsilon}}} = 1$$

f)

$$\int x^2\ln x\,dx = \int\left(\tfrac{1}{3}x^3\right)'\ln x\,dx = \tfrac{1}{3}x^3\ln x - \int \tfrac{1}{3}x^3(\ln x)'\,dx$$

$$= \tfrac{1}{3}x^3\ln x - \int \tfrac{1}{3}x^3\tfrac{1}{x}\,dx = \tfrac{1}{3}x^3\ln x - \tfrac{1}{3}\int x^2\,dx = \tfrac{1}{3}x^3\ln x - \tfrac{1}{9}x^3$$

$$\int_0^1 x^2\ln x\,dx = \lim_{\varepsilon\to 0}\int_\varepsilon^1 x^2\ln x\,dx = \lim_{\varepsilon\to 0}\Big[\tfrac{1}{3}x^3\ln x - \tfrac{1}{9}x^3\Big]_\varepsilon^1$$

$$= \lim_{\varepsilon\to 0}\left(-\tfrac{1}{9} - \tfrac{1}{3}\varepsilon^3\ln\varepsilon + \tfrac{1}{9}\varepsilon^3\right) = -\tfrac{1}{9} - \tfrac{1}{3}\lim_{\varepsilon\to 0}(\varepsilon^3\ln\varepsilon)$$

$$= -\frac{1}{9} - \frac{1}{3}\lim_{\varepsilon\to 0}\frac{\ln\varepsilon}{\frac{1}{\varepsilon^3}} \overset{L'H.}{=} -\frac{1}{9} - \frac{1}{3}\lim_{\varepsilon\to 0}\frac{\frac{1}{\varepsilon}}{-\frac{3}{\varepsilon^4}}$$

$$= -\frac{1}{9} - \frac{1}{3}\lim_{\varepsilon\to 0}\left(-\frac{1}{3}\varepsilon^3\right) = -\frac{1}{9}$$

g)

$$\int \ln(1-x^2)\,dx = (1+x)\ln(1+x) - (1-x)\ln(1-x) - 2x \quad \text{s. Aufgabe 4.6 d)}$$

$$\int_{-1}^{1} \ln(1-x^2)\,dx = \lim_{\varepsilon\to 0}\lim_{\delta\to 0}\int_{-1+\varepsilon}^{1-\delta} \ln(1-x^2)\,dx$$

$$= \lim_{\varepsilon\to 0}\lim_{\delta\to 0}\Big[(1+x)\ln(1+x) - (1-x)\ln(1-x) - 2x\Big]_{-1+\varepsilon}^{1-\delta}$$

$$= \lim_{\varepsilon\to 0}\lim_{\delta\to 0}\Big((2-\delta)\ln(2-\delta) - \delta\ln\delta - 2(1-\delta) - \varepsilon\ln\varepsilon + (2-\varepsilon)\ln(2-\varepsilon) + 2(-1+\varepsilon)\Big)$$

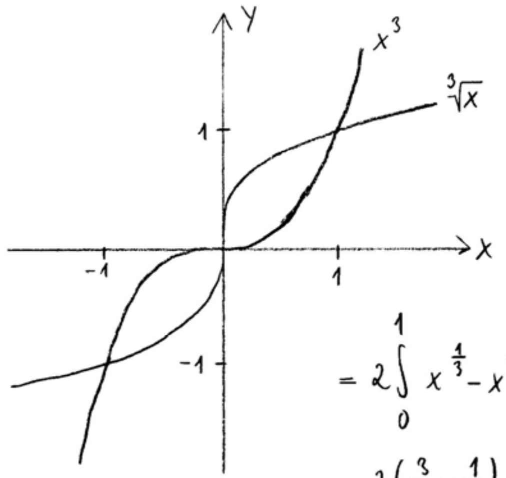
$$= 2\ln 2 - \lim_{\delta\to 0}(\delta\ln\delta) - 2 - \lim_{\varepsilon\to 0}(\varepsilon\ln\varepsilon) + 2\ln 2 - 2$$

$$= 4\ln 2 - 4 - \lim_{\delta\to 0}\frac{\ln\delta}{\frac{1}{\delta}} - \lim_{\varepsilon\to 0}\frac{\ln\varepsilon}{\frac{1}{\varepsilon}}$$

$$\overset{\text{L'H.}}{=} 4\ln 2 - 4 - \lim_{\delta\to 0}\frac{\frac{1}{\delta}}{-\frac{1}{\delta^2}} - \lim \frac{\frac{1}{\varepsilon}}{-\frac{1}{\varepsilon^2}}$$

$$= 4\ln 2 - 4 - \lim_{\delta\to 0}(-\delta) - \lim_{\varepsilon\to 0}(-\varepsilon) = 4\ln 2 - 4$$

4.11



$$\int_{-1}^{1} |f(x) - f^{-1}(x)|\,dx$$

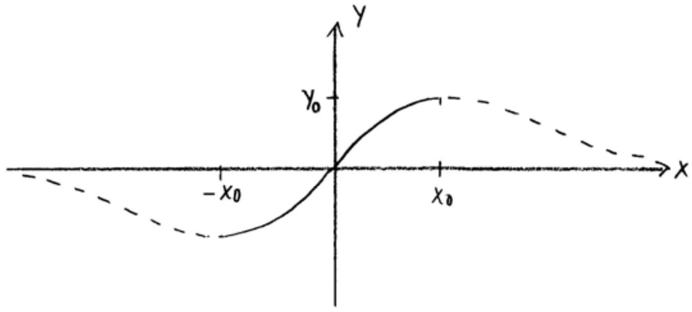
$$= 2\int_0^1 f^{-1}(x) - f(x)\,dx$$

$$= 2\int_0^1 \sqrt[3]{x} - x^3\,dx$$

$$= 2\int_0^1 x^{\frac{1}{3}} - x^3\,dx = 2\left[\frac{3}{4}x^{\frac{4}{3}} - \frac{1}{4}x^4\right]_0^1$$

$$= 2\left(\frac{3}{4} - \frac{1}{4}\right) = 1$$

4.12



$$f(x) = x e^{-x^2} \qquad f'(x) = e^{-x^2} - 2x^2 e^{-x^2} = e^{-x^2}(1-2x^2)$$

$$f'(x) = 0 \Rightarrow 1-2x^2 = 0 \Rightarrow x = \pm \frac{1}{\sqrt{2}} = \pm x_0 \quad \text{mit } x_0 = \frac{1}{\sqrt{2}}$$

$$y_0 = f(x_0) = \frac{1}{\sqrt{2}} e^{-\frac{1}{2}} = \frac{1}{\sqrt{2e}}$$

a)

$$\int f(x)\,dx = \int x e^{-x^2} dx \qquad u = -x^2 \quad \frac{du}{dx} = -2x \quad dx = -\frac{1}{2x} du$$

$$= \int x e^{u} \left(-\frac{1}{2x}\right) du = -\frac{1}{2}\int e^{u} du = -\frac{1}{2} e^{u} = -\frac{1}{2} e^{-x^2}$$

$$\int_0^{x_0} f(x)\,dx = \left[-\frac{1}{2} e^{-x^2}\right]_0^{\frac{1}{\sqrt{2}}} = -\frac{1}{2} e^{-\frac{1}{2}} + \frac{1}{2} = \frac{1}{2}\left(1 - \frac{1}{\sqrt{e}}\right)$$

b)

$$\int_0^{y_0} f^{-1}(y)\,dy \overset{(4.39)}{=} \int_0^{x_0} x f'(x)\,dx = \left[x f(x)\right]_0^{x_0} - \int_0^{x_0} f(x)\,dx$$

$$= x_0 f(x_0) - \int_0^{x_0} f(x)\,dx = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2e}} - \frac{1}{2}\left(1 - \frac{1}{\sqrt{e}}\right) = \frac{1}{2}\frac{1}{\sqrt{e}} - \frac{1}{2} + \frac{1}{2}\frac{1}{\sqrt{e}}$$

$$= \frac{1}{\sqrt{e}} - \frac{1}{2}$$

4.13 a)

$$\int_{x_1}^{x_2} \sqrt{1 + f'(x)^2}\,dx = \int_{-\ln 2}^{\ln 2} \sqrt{1 + \sinh^2 x}\,dx = \int_{-\ln 2}^{\ln 2} \sqrt{\cosh^2 x}\,dx$$

$$= \int_{-\ln 2}^{\ln 2} \cosh x\,dx = 2\int_0^{\ln 2} \cosh x\,dx = 2\left[\sinh x\right]_0^{\ln 2} = 2 \sinh \ln 2$$

$$= 2 \cdot \frac{1}{2}\left(e^{\ln 2} - e^{-\ln 2}\right) = 2 - \frac{1}{2} = \frac{3}{2}$$

b)

$$\int_{x_1}^{x_2}\sqrt{1+f'(x)^2}\,dx = \int_0^1\sqrt{1+4x^2}\,dx = 2\int_0^1\sqrt{(\tfrac{1}{2})^2+x^2}\,dx$$

$$= 2\left[\tfrac{1}{2}x\sqrt{(\tfrac{1}{2})^2+x^2}+\tfrac{1}{2}(\tfrac{1}{2})^2\operatorname{arsinh}\left(\tfrac{x}{\frac{1}{2}}\right)\right]_0^1$$

$$= \sqrt{(\tfrac{1}{2})^2+1}+(\tfrac{1}{2})^2\operatorname{arsinh}2 = \tfrac{1}{2}\sqrt{5}+\tfrac{1}{4}\operatorname{arsinh}2 \approx 1{,}479$$

c)

$$\int_{x_1}^{x_2}\sqrt{1+f'(x)^2}\,dx = \int_0^1\sqrt{1+\left(\tfrac{1}{2\sqrt{x}}\right)^2}\,dx \qquad u=2\sqrt{x}\quad \frac{du}{dx}=\frac{1}{\sqrt{x}}=\frac{2}{u}\quad dx=\frac{1}{2}u\,du$$

$$x_1=0\quad u_1=2\sqrt{x_1}=0\qquad x_2=1\quad u_2=2\sqrt{x_2}=2$$

$$= \frac{1}{2}\int_0^2\sqrt{1+(\tfrac{1}{u})^2}\,u\,du = \frac{1}{2}\int_0^2\sqrt{u^2\left(1+\tfrac{1}{u^2}\right)}\,du = \frac{1}{2}\int_0^2\sqrt{u^2+1}\,du$$

$$= \frac{1}{2}\left[\frac{u}{2}\sqrt{u^2+1}+\frac{1}{2}\operatorname{arsinh}u\right]_0^2 = \frac{1}{2}\left(\sqrt{5}+\frac{1}{2}\operatorname{arsinh}2\right) = \frac{1}{2}\sqrt{5}+\frac{1}{4}\operatorname{arsinh}2$$

d)

$$\int_{x_1}^{x_2}\sqrt{1+f'(x)^2}\,dx = \int_1^2\sqrt{1+(\tfrac{1}{x})^2}\,dx = \int_1^2\sqrt{1+\tfrac{1}{x^2}}\,dx = \int_1^2\sqrt{\tfrac{1}{x^2}(x^2+1)}\,dx$$

$$= \int_1^2\frac{\sqrt{x^2+1}}{x}\,dx = \left[\sqrt{x^2+1}-\ln\frac{1+\sqrt{x^2+1}}{x}\right]_1^2$$

$$= \sqrt{5}-\ln\frac{1+\sqrt{5}}{2}-\sqrt{2}+\ln(1+\sqrt{2})$$

$$= \sqrt{5}-\sqrt{2}+\ln\frac{1+\sqrt{2}}{1+\sqrt{5}}+\ln 2 \approx 1{,}222$$

4.14

$$\int x e^{-\lambda x}\,dx = -\frac{1}{\lambda}\int x\left(e^{-\lambda x}\right)'dx$$

$$= -\frac{1}{\lambda}\left(x e^{-\lambda x}-\int e^{-\lambda x}\,dx\right) = -\frac{1}{\lambda}\left(x e^{-\lambda x}+\frac{1}{\lambda}e^{-\lambda x}\right)$$

$$= -\frac{1}{\lambda}x e^{-\lambda x}-\frac{1}{\lambda^2}e^{-\lambda x} = -\frac{1}{\lambda^2}\,\frac{\lambda x+1}{e^{\lambda x}}$$

$$\int x^2 e^{-\lambda x}\,dx = -\frac{1}{\lambda}\int x^2\left(e^{-\lambda x}\right)'dx$$

$$= -\frac{1}{\lambda}\left(x^2 e^{-\lambda x}-\int 2x e^{-\lambda x}\,dx\right) = -\frac{1}{\lambda}\left(x^2 e^{-\lambda x}-2\int x e^{-\lambda x}\,dx\right)$$

$$= -\frac{1}{\lambda}\left(x^2 e^{-\lambda x}+\frac{2}{\lambda}x e^{-\lambda x}+\frac{2}{\lambda^2}e^{-\lambda x}\right) = -\frac{1}{\lambda^3}\,\frac{\lambda^2x^2+2\lambda x+2}{e^{\lambda x}}$$

$$\lim_{\tau\to\infty}\int_0^{\tau} x e^{\lambda x}\,dx = \lim_{\tau\to\infty}\left[-\frac{1}{\lambda^2}\,\frac{\lambda x+1}{e^{\lambda x}}\right]_0^{\tau} = \lim_{\tau\to\infty}\left(-\frac{1}{\lambda^2}\,\frac{\lambda\tau+1}{e^{\lambda\tau}}+\frac{1}{\lambda^2}\right)$$

$$= \frac{1}{\lambda^2}-\frac{1}{\lambda^2}\lim_{\tau\to\infty}\frac{\lambda\tau+1}{e^{\lambda\tau}} \overset{L'H.}{=} \frac{1}{\lambda^2}-\frac{1}{\lambda^2}\lim_{\tau\to\infty}\frac{\lambda}{\lambda e^{\lambda\tau}} = \frac{1}{\lambda^2}$$

$$\lim_{\tau\to\infty}\int_0^{\tau} x^2 e^{-\lambda x}\,dx = \lim_{\tau\to\infty}\left[-\frac{1}{\lambda^3}\,\frac{\lambda^2x^2+2\lambda x+2}{e^{\lambda x}}\right]_0^{\tau}$$

$$= \lim_{\tau\to\infty}\left(-\frac{1}{\lambda^3}\,\frac{\lambda^2\tau^2+2\lambda\tau+2}{e^{\lambda\tau}}+\frac{2}{\lambda^3}\right)$$

$$= \frac{2}{\lambda^3}-\frac{1}{\lambda^3}\lim_{\tau\to\infty}\frac{\lambda^2\tau^2+2\lambda\tau+2}{e^{\lambda\tau}} \overset{L'H.}{=} \frac{2}{\lambda^3}-\frac{1}{\lambda^3}\lim_{\tau\to\infty}\frac{2\lambda^2\tau+2\lambda}{\lambda e^{\lambda\tau}}$$

$$\overset{L'H.}{=} \frac{2}{\lambda^3}-\frac{1}{\lambda^3}\lim_{\tau\to\infty}\frac{2\lambda^2}{\lambda^2 e^{\lambda\tau}} = \frac{2}{\lambda^3}$$

$$\mu = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^{\infty} x\lambda e^{-\lambda x}\,dx = \lambda\int_0^{\infty} x e^{-\lambda x}\,dx$$

$$= \lambda\lim_{\tau\to\infty}\int_0^{\tau} x e^{-\lambda x}\,dx = \lambda\frac{1}{\lambda^2} = \frac{1}{\lambda}$$

$$\sigma^2 = \int_{-\infty}^{\infty}(x-\mu)^2 f(x)\,dx = \int_0^{\infty}\left(x-\frac{1}{\lambda}\right)^2\lambda e^{-\lambda x}\,dx$$

$$= \int_0^{\infty}\left(x^2-\frac{2}{\lambda}x+\frac{1}{\lambda^2}\right)\lambda e^{-\lambda x}\,dx = \lim_{\tau\to\infty}\int_0^{\tau}\left(\lambda x^2e^{-\lambda x}-2xe^{-\lambda x}+\frac{1}{\lambda}e^{-\lambda x}\right)dx$$

$$= \lim_{\tau\to\infty}\left(\lambda\int_0^{\tau}x^2e^{-\lambda x}\,dx-2\int_0^{\tau}xe^{-\lambda x}\,dx+\frac{1}{\lambda}\int_0^{\tau}e^{-\lambda x}\,dx\right)$$

$$= \lambda\lim_{\tau\to\infty}\int_0^{\tau}x^2e^{-\lambda x}\,dx-2\lim_{\tau\to\infty}\int_0^{\tau}xe^{-\lambda x}\,dx+\frac{1}{\lambda}\lim_{\tau\to\infty}\int_0^{\tau}e^{-\lambda x}\,dx$$

$$= \lambda\frac{2}{\lambda^3}-2\frac{1}{\lambda^2}+\frac{1}{\lambda}\lim_{\tau\to\infty}\left[-\frac{1}{\lambda}e^{-\lambda x}\right]_0^{\tau}$$

$$= \frac{1}{\lambda}\lim_{\tau\to\infty}\left(-\frac{1}{\lambda}e^{-\lambda\tau}+\frac{1}{\lambda}\right) = \frac{1}{\lambda}\lim_{\tau\to\infty}\left(-\frac{1}{\lambda e^{\lambda\tau}}+\frac{1}{\lambda}\right)$$

$$= \frac{1}{\lambda}\cdot\frac{1}{\lambda} = \frac{1}{\lambda^2}$$

$$\sigma = \sqrt{\sigma^2} = \sqrt{\frac{1}{\lambda^2}} = \frac{1}{\lambda}$$

4.15 a)

$$s(t) = \int_0^t v(z)\,dz = \int_0^t gz\,dz = \left[\tfrac{1}{2}gz^2\right]_0^t = \tfrac{1}{2}gt^2$$

b)

$$s(t) = \int_0^t v(z)\,dz = u\int_0^t 1 - e^{-\frac{z}{\tau}}\,dz = u\left[z + \tau e^{-\frac{z}{\tau}}\right]_0^t$$

$$= u\left(t + \tau e^{-\frac{t}{\tau}} - \tau\right)$$

c)

$$s(t) = \int_0^t v(z)\,dz = u\int_0^t \tanh\frac{z}{\tau}\,dz = u\int_0^t \frac{\sinh\frac{z}{\tau}}{\cosh\frac{z}{\tau}}\,dz$$

$$= u\tau\int_0^t \frac{\left(\cosh\frac{z}{\tau}\right)'}{\cosh\frac{z}{\tau}}\,dz = u\tau\left[\ln\cosh\frac{z}{\tau}\right]_0^t = u\tau\ln\cosh\frac{t}{\tau}$$

4.16

$$Q = \sum_{i=1}^{n}\Delta q_i \qquad \Delta q_i \approx qh\frac{r_i}{\sqrt{r_i^2+h^2}^{\,3}}\,\Delta r_i$$

$$Q \approx \sum_{i=1}^{n} qh\frac{r_i}{\sqrt{r_i^2+h^2}^{\,3}}\,\Delta r_i = qh\sum_{i=1}^{n}\frac{r_i}{\sqrt{r_i^2+h^2}^{\,3}}\,\Delta r_i$$

$$Q = qh\lim_{\substack{n\to\infty\\ \Delta r_i\to 0}}\sum_{i=1}^{n}\frac{r_i}{\sqrt{r_i^2+h^2}^{\,3}}\,\Delta r_i = qh\int_0^R\frac{r}{\sqrt{r^2+h^2}^{\,3}}\,dr$$

$$= qh\left[-\frac{1}{\sqrt{r^2+h^2}}\right]_0^R = qh\left(-\frac{1}{\sqrt{R^2+h^2}} + \frac{1}{h}\right)$$

$$= q\left(1 - \frac{h}{\sqrt{R^2+h^2}}\right)$$